ON THE INDENTATION OF A RIGID STAMP INTO AN ANISOTROPIC PLASTIC MEDIUM

(O VDAVLIVANII ZHESTKOGO SHTAMPA V ANIZOTROPNUIU PLASTICHESKUIU SREDU)

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Von Mises [1] was the first to give the yield criterion for an anisotropic material. Later, Hill [2] considered a number of problems in the theory of anisotropic, perfectly plastic bodies. A further paper concerned with this subject is that by Hy [3].

We discuss below the problem of the indentation of a rigid stamp into an anisotropic plastic medium in the case of plane strain. We shall use the yield criterion given in [4].

Let us begin with the basic equations. The equilibrium equations are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{1}$$

The yield criterion is

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2(\theta) \qquad \left(\theta = \frac{1}{2}\tan^{-1}\frac{2\tau_{xy}}{\sigma_y - \sigma_x}\right) \tag{2}$$

and the law of plastic flow is

$$\frac{\varepsilon_x}{k(\sigma_x - \sigma_y) + k'\tau_{xy}} = \frac{\varepsilon_y}{k(\sigma_y - \sigma_x) - k'\tau_{xy}} = \frac{\varepsilon_{xy}}{4k\tau_{xy} - 2k'(\sigma_x - \sigma_y)} \tag{3}$$

where $k' = dk/d\theta$. Using the substitutions

$$\sigma_x = \sigma - k\cos 2\theta, \quad \sigma_y = \sigma + k\cos 2\theta, \quad \tau_{xy} = k\sin 2\theta \tag{4}$$

it is easy to show that Equations (1) and (2) are of the hyperbolic type. The equations of the characteristics are [4]

$$\left(\frac{dy}{dx}\right)_{1,2} = \frac{k' \cos 2\theta - 2k \sin 2\theta \pm \sqrt{k'^2 + 4k^2}}{k' \sin 2\theta + 2k \cos 2\theta} \tag{5}$$

along which we have the integrals

$$\sigma \pm \int \sqrt{k'^2 + 4k^2}\, d\theta = \text{const} \tag{6}$$

The equations for the strain-rate components (3) are of the hyperbolic type and their characteristics are given by Equation (5). Along these characteristics we have the Geiringer relationships

$$du - v d\alpha = 0, \qquad dv + u d\alpha = 0 \tag{7}$$

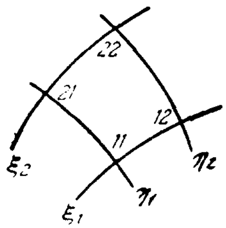

Fig. 1.

which express the absence of elongations along the characteristics, where α is the angle between the characteristics and the x-axis and is given by Equation (5).

From the integrals given by Equation (6) one obtains the formulas which establish the properties of the network of slip lines

$$\sigma_{12} - \sigma_{11} = \sigma_{22} - \sigma_{21}$$

$$\Phi(\theta_{12}) - \Phi(\theta_{11}) = \Phi(\theta_{22}) - \Phi(\theta_{21}) \tag{8}$$

Figure 1 shows the families of slip lines ξ and η.

As a consequence, we find that if a certain section of a slip line ξ is a straight line, then σ, θ and η are constant along it. If both families of slip lines are straight lines then the stresses are uniformly distributed in this region.

If a certain section of a slip line of the family ξ is a straight line, then all the corresponding sections of lines η, cut by the lines ξ, are straight. These properties are a consequence of the analog of the first theorem of Hencky for the form of anisotropy under consideration.

Consider now the problem of the indentation of a stamp as formulated by Hill [2]. We shall seek the solution for bodies having an arbitrary form of anisotropy and assume that the material is homogeneous. The solution will be obtained as a combination of the uniform state-of-stress fields coupled with a centered fan of characteristics as shown in Fig. 2.

Suppose that the normal and shear components of stress σ_n and τ_n are given on the contour. Noting that

$$\sigma_n = \sigma_x \cos^2 \varphi + \sigma_y \sin^2 \varphi + \tau_{xy} \sin 2\varphi, \qquad \tau_n = \frac{1}{2} (\sigma_y - \sigma_x) \sin 2\varphi + \tau_{xy} \cos 2\varphi$$

and using Equation (4), we have

$$\sigma_n = \sigma - k \cos 2 (\theta + \varphi), \ \tau_n = k \sin 2 (\theta + \varphi) \tag{9}$$

where ϕ is the angle between the normal to the contour and the x-axis.

Assuming absence of friction on the surface of contact and that the remaining surface is free of stress, we can use Equation (9) to determine the state of stress in the triangular regions *ABC* and *BDE* of Fig. 2. In the triangle *ABC* we have $\theta = -\pi/2$, and the inclination of the slip lines *ac* is given by Equation (5). The triangle *BDE* will experience a simple compression parallel to the x-axis, and Equation (9) gives $\theta = 0$ and $a = -k(\theta)$. The angles of the triangle can be determined from the equations of the characteristics (5). The hydrostatic pressure σ in the triangle *ABC* along the characteristic *ac* can easily be determined from Equation (6) and is given by

$$\sigma - \int_0^{1/2\pi} \sqrt{k'^2 + 4k^2}\, d\theta = -k(0)$$

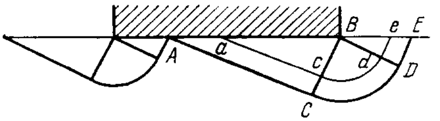

Fig. 2.

The stress components in the triangle *ABC* can be found from Equation (4) and are given by

$$\sigma_x = -k(0) + \int_0^{-1/2\pi} \sqrt{k'^2 + 4k^2}\, d\theta + k(-1/2\pi)$$

$$p = \sigma_y = -k(0) - k(-1/2\pi) + \int_0^{1/2\pi} \sqrt{k'^2 + 4k^2}\, p\theta$$

The latter formula determines the limiting pressure p of the piston on the surface of an anisotropic, perfectly plastic medium. It follows from the above formulas that the limit load depends on the form of the functions $k(\theta)$ in the range $-\pi/2 \leqslant \theta \leqslant 0$. In the case of an isotropic material, the Prandtl formula is automatically satisfied.

Consider now the distribution of the rates of displacement. The triangle *ABC* slides as a solid body along *AC* with a velocity $V/\sin a_1$, where V is the rate at which the stamp is pressed in and a_1 is the angle *BAC*. In the centered field *BCD*, the velocity is $V/\sin a_1$ along *cd* and zero in the perpendicular direction. The triangle *BDE* moves in the direction of the line *de* with a velocity $V/\sin a_1$.

The distribution of stress and velocities to the left of the point *A* can be obtained in a similar way. Clearly, in the case of an isotropic material, the distribution of displacement velocities will be the same as in the Hill solution [2].

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